

積分法の基礎 第①講 第②講

例題1 次の積分を計算しなさい。

$$(1) \int \sqrt{x}(x-1)dx$$

$$(2) \int \sqrt{x+1}(1-x)dx$$

$$(3) \int \frac{dx}{x^2-1}$$

$$(4) \int \frac{x^4+3x^2-x}{x^3-1}dx$$

$$(5) \int \sin^3x dx$$

$$(6) \int \frac{\sin^2x}{\cos^4x} dx$$

$$(7) \int \frac{(\log x)^2}{x} dx$$

$$(8) \int \sin mx \cos nx dx$$

演習①

$$(1) \int \frac{dx}{e^x+1}$$

$$(2) \int e^{-\sqrt{x}} dx$$

$$(3) \int \frac{x}{\cos^2x} dx$$

$$(4) \int \log(x+\sqrt{1+x^2})dx$$

$$(5) \int x(\log x)^2 dx$$

例題2 次の等式を証明しなさい。

$$\int_0^\pi xf(\sin x)dx = \frac{\pi}{2} \int_0^\pi f(\sin x)dx$$

上の等式を利用して、次の積分を計算しなさい。

$$I = \int_0^\pi \frac{x \sin x}{3 + \sin^2 x} dx$$

演習②

$$(2) I = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x} dx$$

$$(2) I = \int_0^{\frac{\pi}{4}} \log(1 + \tan x) dx$$

例1

(1) $\int \sqrt{x}(x-1) dx = \int (x^{\frac{3}{2}} - x^{\frac{1}{2}}) dx = \frac{2}{5} x^{\frac{5}{2}} - \frac{2}{3} x^{\frac{3}{2}} + C //$

(2) $\int \sqrt{x+1}(1-x) dx = \int \sqrt{x+1}(2-(x+1)) dx = 2 \int (x+1)^{\frac{1}{2}} dx - \int (x+1)^{\frac{3}{2}} dx$
 $= \frac{4}{3} (x+1)^{\frac{3}{2}} - \frac{2}{5} (x+1)^{\frac{5}{2}} + C //$

(3) $\int \frac{1}{x^2-1} dx = \frac{1}{2} \int \left(\frac{1}{x-1} - \frac{1}{x+1} \right) dx = \frac{1}{2} (\log|x-1| - \log|x+1|) + C$
 $= \frac{1}{2} \log \left| \frac{x-1}{x+1} \right| + C //$

(4) $\int \frac{x^4+3x^2-x}{x^3-1} dx = \int \frac{x(x^2-1)+3x^2}{x^3-1} dx = \int \left(x + \frac{3x^2}{x^3-1} \right) dx = \int x dx + \int \frac{(x^3-1)'}{x^3-1} dx$
 $= \frac{1}{2} x^2 + \log|x^3-1| + C //$

(5) $\int \sin^3 x dx = \int \left(\frac{3}{4} \sin x - \frac{1}{4} \sin 3x \right) dx = -\frac{3}{4} \cos x + \frac{1}{12} \cos 3x + C //$

(6) $\int \frac{\sin^2 x}{\cos^4 x} dx = \int \frac{\sin^2 x}{\cos^2 x} \cdot \frac{1}{\cos^2 x} dx = \int \tan^2 x \cdot (\tan x)' dx = \frac{1}{3} \tan^3 x + C //$

(7) $\int \frac{(\log x)^2}{x} dx = \int (\log x)^2 \cdot (\log x)' dx = \frac{1}{3} (\log x)^3 + C //$

(8) $\int \sin mx \cdot \cos nx dx$

i) $m \neq n$ のとき

$\sin mx \cos nx = \frac{1}{2} \{ \sin(m+n)x + \cos(m-n)x \}$ ④

④式 $= \frac{1}{2(m+n)} \{ -\cos(m+n)x \} + \frac{1}{2} \frac{1}{m-n} \sin(m-n)x + C$

$= \frac{1}{2} \left\{ \frac{1}{m-n} \sin(m-n)x - \frac{1}{m+n} \cos(m+n)x \right\} + C //$

ii) $m = n$ のとき

④式 $= \frac{1}{2} \int \sin 2mx dx = -\frac{1}{4m} \cos 2mx + C //$

演習1.

(1) $\int \frac{1}{e^x+1} dx = \int \frac{e^x}{1+e^x} = \int \frac{-(1+e^x)'}{1+e^x} dx = -\log(1+e^x) + C //$

※ ④の形を e^x で割ると $\frac{1}{1+e^x} \rightarrow \frac{e^x}{1+e^x}$ の形が利用可能。

(2) $\int e^{-\sqrt{x}} dx = \int e^t \cdot 2t dt = 2 \int t \cdot (e^t)' dt$
 $= 2t \cdot e^t - 2 \int e^t dt$
 $= 2t \cdot e^t - 2e^t + C$
 $= 2e^t(t-1) + C$
 $= 2e^{-\sqrt{x}}(-\sqrt{x}-1) + C$
 $= -2e^{-\sqrt{x}}(\sqrt{x}+1) + C //$

$$\square (3) \int \frac{x}{\cos^2 x} dx = \int (\tan x)' \cdot x dx = x \cdot \tan x - \int \tan x dx \\ = x \tan x - \log |\cos x| + C //$$

$$(4) \int \log(x + \sqrt{1+x^2}) dx$$

$$x = \frac{e^t - e^{-t}}{2} \text{ とき } dx = \frac{e^t + e^{-t}}{2} dt$$

$$\text{与式} = \int \log\left(\frac{e^t - e^{-t}}{2} + \sqrt{1 + \left(\frac{e^t - e^{-t}}{2}\right)^2}\right) \cdot \frac{e^t + e^{-t}}{2} dt = \int \log\left(\frac{e^t - e^{-t}}{2} + \frac{e^t + e^{-t}}{2}\right) \cdot \frac{e^t + e^{-t}}{2} dt$$

$$= \int \log e^t \cdot \frac{e^t + e^{-t}}{2} dt = \frac{1}{2} \int t(e^t + e^{-t}) dt = \frac{1}{2} \int t \cdot (e^t - e^{-t})' dt$$

$$= \frac{1}{2} \left\{ t(e^t - e^{-t}) - \int (e^t - e^{-t}) dt \right\} = \frac{1}{2} \left\{ t(e^t - e^{-t}) - e^t + e^{-t} \right\} + C \dots (A)$$

$$\therefore \because 2x = e^t - e^{-t} \rightarrow (e^t)^2 - 2x \cdot (e^t) - 1 = 0 \rightarrow e^t = x \pm \sqrt{x^2 + 1}$$

$$e^t > 0 \text{ 故 } e^t = x + \sqrt{x^2 + 1}, e^{-t} = \frac{1}{x + \sqrt{x^2 + 1}} = \sqrt{x^2 + 1} - x$$

$$\therefore e^t + e^{-t} = 2\sqrt{x^2 + 1}, e^t - e^{-t} = 2x \quad \left\{ \text{ (A) を代} \right\}$$

$$t = \log(x + \sqrt{x^2 + 1})$$

$$= \frac{1}{2} \left\{ \log(x + \sqrt{x^2 + 1}) \right\} \cdot 2x - 2\sqrt{x^2 + 1} \quad \left\{ + C \right.$$

$$= x \log(x + \sqrt{x^2 + 1}) - \sqrt{x^2 + 1} + C //$$

$$(5) \int x (\log x)^2 dx = \int \left(\frac{1}{2}x^2\right)' (\log x)^2 dx = \frac{1}{2}x^2 (\log x)^2 - \int \left(\frac{1}{2}x^2\right) \cdot 2 \log x \cdot \left(\frac{1}{x}\right) dx$$

$$= \frac{1}{2}x^2 (\log x)^2 - \int x \log x dx$$

$$= \frac{1}{2}x^2 (\log x)^2 - \left\{ \left(\frac{1}{2}x^2\right) \log x - \int \frac{1}{2}x^2 \cdot \frac{1}{x} dx \right\}$$

$$= \frac{1}{2}x^2 (\log x)^2 - \frac{1}{2}x^2 \log x + \frac{1}{2} \int x dx$$

$$= \frac{1}{2}x^2 (\log x)^2 - \frac{1}{2}x^2 \log x + \frac{1}{4}x^2 + C //$$

例2 $\int_0^\pi x f(\sin x) dx = \frac{\pi}{2} \int_0^\pi f(\sin x) dx$ の証明

$$\begin{aligned} x = \pi - t \text{ とおくと} & \quad \begin{array}{l} x \mid 0 \rightarrow \pi \\ t \mid \pi \rightarrow 0 \end{array} \\ dx = -dt & \\ \text{左辺} = \int_\pi^0 (\pi - t) f(\sin(\pi - t)) (-dt) &= \int_0^\pi (\pi - t) f(\sin t) dt \quad (\because \sin(\pi - t) = \sin t) \\ &= \pi \int_0^\pi f(\sin t) dt - \int_0^\pi t f(\sin t) dt \end{aligned}$$

$$\begin{aligned} \text{よって } I = \pi \int_0^\pi f(\sin t) dt - I &\rightarrow 2I = \pi \int_0^\pi f(\sin t) dt = \pi \int_0^\pi f(\sin x) dx \\ \therefore \text{左辺} = I &= \frac{1}{2} \pi \int_0^\pi f(\sin x) dx \dots (A) \end{aligned}$$

$$\therefore K = \int_0^\pi \frac{x \sin x}{3 + \sin^2 x} dx = \frac{\pi}{2} \int_0^\pi \frac{\sin x}{3 + \sin^2 x} dx = \frac{\pi}{2} \int_0^\pi \frac{\sin x}{3 + (1 - \cos^2 x)} dx //$$

$$\begin{aligned} \left. \begin{array}{l} \cos x = t \text{ とおくと} \\ -\sin x dx = dt \\ \begin{array}{l} x \mid 0 \rightarrow \pi \\ t \mid 1 \rightarrow 0 \end{array} \end{array} \right\} \begin{aligned} K &= \frac{\pi}{2} \int_1^0 \frac{-dt}{4 - t^2} = \frac{\pi}{2} \int_0^1 \left(\frac{1}{2-t} + \frac{1}{2+t} \right) \times \frac{1}{4} dt \\ &= \frac{\pi}{8} \int_0^1 \left(\frac{1}{t+2} - \frac{1}{t-2} \right) dt \\ &= \frac{\pi}{8} \left[\log \left| \frac{t+2}{t-2} \right| \right]_0^1 = \frac{\pi}{8} (\log 3 - 0) = \frac{\pi}{8} \log 3 // \end{aligned}$$